

ON DISTORTION IN GROUPS OF HOMEOMORPHISMS

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ABSTRACT. Let X be a path-connected topological space admitting a universal cover. Let $\text{Homeo}(X, \alpha)$ denote the group of homeomorphisms of X preserving degree one cohomology class α .

We investigate the distortion in $\text{Homeo}(X, \alpha)$. Let g be an element of $\text{Homeo}(X, \alpha)$. We define a Nielsen-type equivalence relation on the space of g -invariant Borel probability measures on X and prove that if a homeomorphism g admits two nonequivalent invariant measures then it is undistorted. We also define a local rotation number of a homeomorphism generalising the notion of the rotation of a homeomorphism of the circle. Then we prove that a homeomorphism is undistorted if its rotation number is nonconstant.

1. INTRODUCTION AND THE STATEMENT OF THE RESULTS

Let X be a path-connected topological space admitting a universal cover. Let $\text{Homeo}(X, \alpha)$ denote the group of homeomorphisms of X preserving a cohomology class $\alpha \in H^1(X; \mathbf{R})$.

In the present paper, we study distortion in $\text{Homeo}(X, \alpha)$. We define the distortion in groups and briefly discuss this notion in Section [1.A](#).

Let $g \in \text{Homeo}(X, \alpha)$. By α we denote a singular one-cocycle representing the class α . Let $\mathfrak{K}_\alpha(g) : X \rightarrow \mathbf{R}$ be a function such that

$$\delta \mathfrak{K}_\alpha(g) = g^* \alpha - \alpha,$$

where δ denotes the codifferential map $C^0(X; \mathbf{R}) \rightarrow C^1(X, \mathbf{R})$ between singular zero-cocycles (functions) and one-cocycles on X . Notice that $\mathfrak{K}_\alpha(g)$ exists since $g^* \alpha$ and α are in the same cohomology class. In other terms a function $\mathfrak{K}_\alpha(g)$ is defined by the condition that for any $x, y \in X$ one has $\mathfrak{K}_\alpha(g)(y) - \mathfrak{K}_\alpha(g)(x) = \int_\gamma (g^* \alpha - \alpha)$, where γ denotes any path between x and y , and the expression $\int_\gamma \sigma$ denotes the natural pairing of a chain γ and a cochain σ . We find this nonstandard notation useful because we would like to think that the cocycle α is defined by the integration of a differential form over smooth paths.

Key words and phrases. distortion in groups; rotation number; groups of homeomorphisms; invariant measures.

We discuss the function $\mathfrak{K}_\alpha(g)$ in Section 2.A where we prove that it is continuous for a suitable choice of the cocycle α . In particular, the function $\mathfrak{K}_\alpha(g)$ is integrable with respect to any Borel probability measure on X . If not stated otherwise, we assume that α is chosen in such a way.

Assume that the homeomorphism $g \in \text{Homeo}(X, \mathfrak{a})$ admits two invariant Borel probability measures μ and ν . We say that μ and ν are **α -Nielsen equivalent** (we will motivate the name after the proof of Corollary 1.2) if

$$\int \mathfrak{K}_\alpha(g) \mu = \int \mathfrak{K}_\alpha(g) \nu.$$

It is clear that $\mathfrak{K}_\alpha(g)$ is defined up to an additive constant, but the difference $\int \mathfrak{K}_\alpha(g) \mu - \int \mathfrak{K}_\alpha(g) \nu$ does not depend on this choice. If the class \mathfrak{a} is fixed we will simply call measures Nielsen equivalent.

The following theorem is proven in Section 2.C.

Theorem 1.1. *Let X be compact and let $g \in \text{Homeo}(X, \mathfrak{a})$. Assume that g admits invariant Nielsen nonequivalent measures then g is undistorted in $\text{Homeo}(X, \mathfrak{a})$.*

The first corollary of the above theorem is when the measures are supported on two fixed points of g .

Corollary 1.2. *Let X be compact and let $g \in \text{Homeo}(X, \mathfrak{a})$ where $\mathfrak{a} \in H^1(X; \mathbb{R})$ is represented by a one-cocycle α . Suppose that g has two fixed points $x, y \in X$ and let γ be a path from x to y . If*

$$\langle \mathfrak{a}, g\gamma - \gamma \rangle \neq 0$$

then g is undistorted in $\text{Homeo}(X, \mathfrak{a})$.

Proof. We have to check that atomic measures δ_x and δ_y supported on x and y are not Nielsen equivalent. By the definition of $\mathfrak{K}_\alpha(g)$ we have that

$$\int \mathfrak{K}_\alpha(g)(\delta_y - \delta_x) = \mathfrak{K}_\alpha(g)(y) - \mathfrak{K}_\alpha(g)(x) = \int_Y g^* \alpha - \alpha = \langle \mathfrak{a}, g\gamma - \gamma \rangle \neq 0.$$

□

Two fixed points x, y of a map $g: X \rightarrow X$ are called **Nielsen equivalent** if there exists a path γ from x to y such that γ and $g\gamma$ are homotopic modulo the endpoints. The hypothesis of the above theorem implies that the homeomorphism g has two fixed points which are Nielsen nonequivalent in a stronger sense. Namely, the cycle $g\gamma - \gamma$ is homologically nontrivial.

Example 1.3. Let $G \subset \text{Homeo}(\Sigma)$ be a group of homeomorphisms of a closed oriented surface Σ acting trivially on the first cohomology of Σ . Suppose that $g \in G$ has two fixed points $x, y \in \Sigma$ such that $g\gamma - \gamma$ is a homologically nontrivial loop, where γ is a path from x to y . Then g is undistorted in G . Indeed, there exists a cohomology class $\alpha \in H^1(\Sigma; \mathbf{Z})$ evaluating nontrivially on $g\gamma - \gamma$. \diamond

Another instance where Theorem 1.1 applies is the following.

Corollary 1.4. *Let $X = \mathbf{S}^1 \times [0, 1]$ be the closed annulus and let $h: X \rightarrow X$ be a homeomorphism preserving the orientation and the components of the boundary. If the topological rotation numbers of h restricted to the boundary circles are distinct then invariant measures supported on boundary circles are Nielsen nonequivalent. Thus, by Theorem 1.1, h is undistorted in the group of orientation preserving homeomorphisms of X .*

This is a corollary of a more general statement (proven in Section 2.D).

Proposition 1.5. *Assume that $\alpha \in H^1(X, \mathbf{Z})$. Let ℓ_1 and ℓ_2 be simple closed curves invariant by a homeomorphism $g \in \text{Homeo}(X, \alpha)$. Let $\rho_i \in \mathbf{R}/\mathbf{Z}$ denote the topological rotation number of g on ℓ_i . If $\rho_1 \langle \alpha, \ell_1 \rangle \neq \rho_2 \langle \alpha, \ell_2 \rangle$ then (any) g -invariant measures supported on ℓ_1 and ℓ_2 are Nielsen nonequivalent.*

Observe that the expression from defining \mathfrak{K}_α leads to a definition of a two-cocycle on the group $\text{Homeo}(X, \alpha)$ with trivial coefficients. Let $g, h \in \text{Homeo}(X, \alpha)$ and let $\gamma: [0, 1] \rightarrow X$ be a continuous path from a reference point $x \in X$ to its image hx . Define

$$\mathfrak{G}_{x, \alpha}(g, h) := \int_\gamma g^* \alpha - \alpha,$$

where α is a singular one-cocycle representing the class α .

In Section 3.C we consider the case when $\alpha \in H^1(X, \mathbf{Z})$ and α is an integer valued one-cocycle representing α . In general, the two-cocycle $\mathfrak{G}_{x, \alpha}$ is not a bounded cocycle, but in Section 3.C we define a local rotation number of a homeomorphism g with respect to a point x as $[\mathfrak{G}_{x, \alpha}] \in H_b^2(\mathbf{Z}; \mathbf{Z}) = \mathbf{R}/\mathbf{Z}$ provided the cocycle $\mathfrak{G}_{x, \alpha}$ is a bounded two-cocycle on the cyclic group generated by g .

If $X = \mathbf{S}^1$ then the two-cocycle $\mathfrak{G}_{x, \alpha}$ corresponding to the length form is the (well studied) Euler cocycle. In particular, the local rotation number equals the classical topological rotation number of an orientation preserving homeomorphism of the circle [11, Section 6.3]. More precisely, if ℓ is an invariant circle in X then the local rotation number of a point in ℓ equals to the topological rotation number of the action on ℓ times $\langle \alpha, \ell \rangle$.

The points where $\mathfrak{G}_{x,\alpha}$ is a bounded cocycle can be used to detect nondistortion instead of invariant measures as the following result shows (which is a consequence of a more general result, Theorem 3.5, proven in Section 3.B).

Theorem 1.6. *Let $\alpha \in H^1(X; \mathbf{Z})$ and let $g \in \text{Homeo}(X, \alpha)$ and assume that X is compact. Let x and y be points such that the cocycles $\mathfrak{G}_{x,\alpha}$ and $\mathfrak{G}_{y,\alpha}$ are bounded on the cyclic subgroup generated by g . If the local rotation numbers of g at x and y are distinct then g is undistorted in $\text{Homeo}(X, \alpha)$.*

Notice that Theorem 1.6 implies, in particular, Proposition 1.5.

Example 1.7. Let X be a closed oriented manifold with non-zero Euler characteristic and with positive first Betti number (e.g. a surface of genus at least two). Let $F: \tilde{X} \rightarrow \mathbf{R}$ be a function such that $dF = p^* \alpha$, where $p: \tilde{X} \rightarrow X$ is the universal cover and α is a closed one-form on X with integral periods representing a nonzero cohomology class $\alpha \in H^1(X; \mathbf{Z})$.

Let $g \in \text{Homeo}(X, \alpha)$ be a homeomorphism and let $\tilde{g} \in \text{Homeo}(\tilde{X})$ denote its lift. If a point $\tilde{x} \in \tilde{X}$ is such that the following limit

$$\lim_{n \rightarrow \infty} \frac{F(\tilde{g}^n(\tilde{x}))}{n}$$

exists and is not an integer then g is undistorted in $\text{Homeo}(X, \alpha)$. Indeed, since the Euler characteristic of X is nonzero the homeomorphism g has a fixed point $y \in X$. The local rotation number of a fixed point is equal to zero. On the other hand the local rotation number of $x := p(\tilde{x})$ is equal modulo integers and up to a sign to the above limit (see Proposition 3.10) and it follows from the above assumption that it is nonzero. \diamond

1.A. Distortion in groups. Let Γ be a finitely generated group. Define the word norm associated with fixed set of generators S to be

$$|g| := \min\{k \in \mathbf{N} \mid g = s_1 \dots s_k, s_i \in S\}.$$

The **translation length** of an element $g \in \Gamma$ is defined to be

$$\tau(g) := \lim_{n \rightarrow \infty} \frac{|g^n|}{n}.$$

An element $g \in \Gamma$ is called **undistorted** if its translation length is positive and this property does not depend on the choice of generators. If G is a general (not necessarily finitely generated) group then $g \in G$ is called **undistorted** if it is undistorted in every finitely generated subgroup of G . Notice that distortion in a subgroup implies distortion in the ambient group.

The distortion is a tool in understanding group actions on manifolds. For example, it is well known that certain lattices in semisimple Lie groups contain distorted elements due to a result of Lubotzky-Mozes and Raghunathan [14]. On the other hand, the distortion in groups of diffeomorphisms of closed manifolds is rare as shown, for example, by Franks and Handel [6], Gambaudo and Ghys [10], or Polterovich [16]. This provides restrictions on possible actions of such lattices.

The papers cited above are concerned with the distortion either in volume preserving or in Hamiltonian diffeomorphisms. It follows from our results, however, that many elements are undistorted in groups of homeomorphisms of manifolds of dimension at least two and with nontrivial first real cohomology. Essentially, this is as much as one gets for such manifolds. In contrast, Calegari and Freedman [4, Theorem C] proved that all homeomorphisms of the sphere \mathbf{S}^n are distorted in $\text{Homeo}(\mathbf{S}^n)$.

Historical remarks. The cocycle $\mathfrak{G}_{x,\alpha}$ can be defined for an arbitrary, not necessarily closed, one-cochain α on a suitably defined subgroup of the group $\text{Homeo}(X)$. It has been first defined by Ismagilov, Losik, and Michor in [13] for a primitive of a symplectic form and further studied by the authors in [8].

The cocycle \mathfrak{K}_α (see Section 2.A for definition) appears in Gambaudo and Ghys [9] and in Arnold and Khesin [1, p. 247] in the case of a symplectic ball. It has been studied for a general symplectically aspherical manifold in [7].

The local rotation number generalizes the rotation number of a homeomorphism of a circle. There are related notions in the literature. For example the rotation vector of a surface diffeomorphism defined by Franks in [5, Definition 2.1], or the rotation defined by Burger, Iozzi, and Wienhard in [2, Definition 7.1].

2. PROOFS OF MAIN RESULTS

2.A. The one-cocycle \mathfrak{K}_α . If $g \in G \subset \text{Homeo}(X, \alpha)$ then $g^*\alpha - \alpha$ is an exact singular one-cocycle on X and the identity $\delta(\mathfrak{K}_\alpha(g)) = g^*\alpha - \alpha$ defines a map

$$\mathfrak{K}_\alpha: G \rightarrow C^0(X; \mathbf{R})/\mathbf{R}.$$

It is straightforward to check that \mathfrak{K}_α is a one-cocycle (cf. [7, Proposition 2.3]). That is, it satisfies

$$\mathfrak{K}_\alpha(gh) = \mathfrak{K}_\alpha(g) \circ h + \mathfrak{K}_\alpha(h)$$

for all $g, h \in G$.

Lemma 2.1. *Assume that X is paracompact. Let $\mathfrak{a} \in H^1(X; \mathbf{R})$. There exists a singular cocycle α representing the class \mathfrak{a} such that for any homeomorphism $g \in \text{Homeo}(X, \mathfrak{a})$ the function $\mathfrak{K}_\alpha(g)$ is a continuous function.*

Remark 2.2. If X is a differentiable manifold and then every real cohomology class is represented by a smooth and closed differential form α . It follows that for any diffeomorphism $h \in \text{Diff}(X, \mathfrak{a})$ the function $\mathfrak{K}_\alpha(h)$ is smooth.

Proof of Lemma 2.1. Let us consider the real numbers \mathbf{R} endowed with the usual order topology and consider the bundle

$$\mathbf{R} \rightarrow E = \tilde{X} \times_{\pi_1 X} \mathbf{R} \xrightarrow{p} X.$$

Since the fibre is contractible and the base is paracompact it admits a continuous section $s: X \rightarrow E$. Such a section defines a continuous equivariant function $\mathbf{a}: E \rightarrow \mathbf{R}$ by the identity $p[\tilde{x}, t] = \mathbf{a}[\tilde{x}, t] + sp[\tilde{x}, t]$. The equivariance means that $\mathbf{a}[\tilde{x}, t + s] = \mathbf{a}[\tilde{x}, t] + s$.

Let $X_\mathfrak{a} = \tilde{X} \times_{\pi_1 X} \mathbf{R}^\delta$ be a covering associated with the class \mathfrak{a} , where \mathbf{R}^δ denotes the real numbers equipped with the discrete topology. Observe that $X_\mathfrak{a}$ is equal to E as a set but it has a finer topology. Thus $\mathbf{a}: X_\mathfrak{a} \rightarrow \mathbf{R}$ is still a continuous function.

Let $\tilde{g} \in \text{Homeo}(X_\mathfrak{a})$ be an \mathbf{R} -equivariant lift of $g \in \text{Homeo}(X, \mathfrak{a})$. Define a continuous function $\hat{\mathfrak{K}}(g): X_\mathfrak{a} \rightarrow \mathbf{R}$ by

$$\hat{\mathfrak{K}}(g)[\tilde{x}, t] := \mathbf{a}(\tilde{g}[\tilde{x}, t]) - \mathbf{a}[\tilde{x}, t].$$

Since both \tilde{g} and \mathbf{a} are \mathbf{R} -equivariant the function $\hat{\mathfrak{K}}(g)$ is \mathbf{R} -invariant and thus descends to a continuous function $\mathfrak{K}(g): X \rightarrow \mathbf{R}$.

Let us show that $\mathfrak{K} = \mathfrak{K}_\alpha$. Let γ be a path between x and y . Let $\tilde{\gamma}$ be its lift with endpoints at \tilde{x} and \tilde{y} . Then

$$\begin{aligned} \mathfrak{K}(g)(y) - \mathfrak{K}(g)(x) &= (\mathbf{a}(\tilde{g}\tilde{y}) - \mathbf{a}(\tilde{g}\tilde{x})) - (\mathbf{a}(\tilde{y}) - \mathbf{a}(\tilde{x})) \\ &= \int_{g\gamma} \alpha - \int_\gamma \alpha = \int_\gamma g^* \alpha - \alpha. \end{aligned}$$

The second equality above follows from the bijective correspondence between singular one-cocycles and \mathbf{R} -equivariant functions $X_\mathfrak{a} \rightarrow \mathbf{R}$ up to the constants. For the convenience of the reader we explain this folklore fact in Section 4. \square

2.B. A seminorm on $\text{Homeo}(X, \mathfrak{a})$. Let X be a compact space. Let us define a seminorm of an element $g \in \text{Homeo}(X, \mathfrak{a})$ by

$$\|g\|_\alpha := \sup_{x, y \in X} |\mathfrak{K}_\alpha(g)(y) - \mathfrak{K}_\alpha(g)(x)|.$$

This means that $\|\cdot\|_\alpha$ is symmetric and satisfies the triangle inequality. The finiteness of $\|g\|_\alpha$ is a consequence of the compactness of X according to Lemma 2.1. It follows that if $\Gamma \subset \text{Homeo}(X, \mathfrak{a})$ is a subgroup generated by a finite set S then

$$C \cdot |g| \geq \|g\|_\alpha,$$

where $C := \max\{\|s\|_\alpha \mid s \in S\}$ and $|g|$ denotes the word norm of $g \in \Gamma$. This is just a special case of the standard and straightforward to prove fact that any seminorm on a group is Lipschitz with respect to the word norm.

2.C. Proof of Theorem 1.1. Let g be homeomorphisms of X preserving the class \mathfrak{a} and Borel probability measures μ and ν . Recall that we need to show that g is undistorted in $\text{Homeo}(X, \mathfrak{a})$ if μ and ν are Nielsen nonequivalent.

Let $\Gamma \subset \text{Homeo}(X, \mathfrak{a})$ be an arbitrary finitely generated group containing g . As we mentioned above its inclusion is Lipschitz with respect to the word metric $|\cdot|$ on Γ and the seminorm $\|\cdot\|_\alpha$ on $\text{Homeo}(X, \mathfrak{a})$.

Observe that the map defined by

$$\text{Homeo}(X, \mathfrak{a}) \ni h \mapsto \int \mathfrak{K}_\alpha(h)(\mu - \nu) \in \mathbf{R}$$

is one-Lipschitz with respect to the seminorm $\|\cdot\|_\alpha$ and it is a homomorphism on the cyclic group generated by g (in fact, on a group of homeomorphisms preserving \mathfrak{a} as well as measures μ and ν). From this we get the following estimate of the word norm of g .

$$\begin{aligned} C \frac{|g^n|}{n} &\geq \frac{\|g\|_\alpha}{n} \\ &\geq \frac{1}{n} \cdot \left| \int \mathfrak{K}_\alpha(g^n)(\mu - \nu) \right| \\ &= \left| \int \mathfrak{K}_\alpha(g)(\mu - \nu) \right| > 0 \end{aligned}$$

This shows that the translation length of g in Γ is positive. Since Γ is an arbitrary finitely generated subgroup of $\text{Homeo}(X, \mathfrak{a})$, this proves that g is undistorted in $\text{Homeo}(X, \mathfrak{a})$. \square

Remark 2.3. The above proof is essentially the same as the proof of the Polterovich theorem [16, Section 5.3] about the distortion in the group of Hamiltonian diffeomorphisms of a closed symplectically hyperbolic manifold presented by the authors in [7].

2.D. Proof of Proposition 1.5. Recall that we need to prove that given simple curve closed curves ℓ_1 and ℓ_2 invariant by a homeomorphism $g \in \text{Homeo}(X, \alpha)$ if $\rho_1 \langle \alpha, \ell_1 \rangle \neq \rho_2 \langle \alpha, \ell_2 \rangle$ then g -invariant measures supported on ℓ_i and Nielsen nonequivalent.

Let α represent α . It is clear that if μ_i denote any invariant measure supported on ℓ_i then

$$\int_{\ell_i} \left(\int_x^{gx} \alpha \right) \mu(dx) = \rho_i \int_{\ell_i} \alpha = \rho_i \langle \alpha, \ell_i \rangle.$$

Notice that inner integral depends on the choice of curves between x and gx but once such choice is made depending continuously on x the value of the integral modulo integers would not depend on that choice.

Let γ be a path between x and y , and let η_x and η_y be paths between x and gx and y and gy respectively. Let us choose η_y to be a concatenation of $-\gamma$, η_x , and $g\gamma$. Then $\int_{\eta_x} \alpha + \int_{\gamma} g^* \alpha - \int_{\eta_y} \alpha - \int_{\gamma} \alpha = 0$. This can be rewritten as

$$\mathfrak{K}_\alpha(g)(y) - \mathfrak{K}_\alpha(g)(x) = \int_{\eta_y} \alpha - \int_{\eta_x} \alpha.$$

Averaging the above equality over x and y with respect to μ_1 and μ_2 respectively we get

$$\int \mathfrak{K}_\alpha(g)(\mu_1 - \mu_2) = \rho_1 \int_{\ell_1} \alpha - \rho_2 \int_{\ell_2} \alpha \neq 0.$$

This proves the claim. \square

Corollary 2.4. *Let $G \subset \text{Homeo}(X)$ be a group of homeomorphisms acting trivially on $H^1(X; \mathbf{R})$. Let g be a homeomorphism distorted in G . Let $\ell_1, \ell_2 \subset X$ be oriented simple closed curves preserved by g . Assume also that the classes $[\ell_i]$ are nonzero in $H^1(X; \mathbf{R})$. Let ρ_1 and ρ_2 be the topological rotation numbers associated with the action of g on ℓ_1 and ℓ_2 respectively. Then the following statements hold:*

- (1) *There exist nonzero integers $k_1, k_2 \in \mathbf{Z}$ such that $k_1 \rho_1 = k_2 \rho_2$.*
- (2) *If, moreover, the classes $[\ell_1]$ and $[\ell_2]$ in $H^1(X; \mathbf{R})$ are linearly independent then $\rho_1, \rho_2 \in \mathbf{Q}/\mathbf{Z}$.*

Proof. Indeed, Let α be an one-cocycle with integral periods.

Choosing α such that $\int_{\ell_i} \alpha \neq 0$ proves the first statement.

To prove the second assertion, we choose α such that $\int_{\ell_1} \alpha = 0 \neq \int_{\ell_2} \alpha$. It follows that $\rho_2 \cdot \int_{\ell_2} \alpha = 0$ and, since $\int_{\ell_2} \alpha$ is an integer, it implies that $\rho_2 \in \mathbf{Q}/\mathbf{Z}$. The rationality of ρ_1 is proved similarly. \square

3. FURTHER RESULTS

3.A. The cocycle $\mathfrak{G}_{x,\alpha}$. Recall that X is a path-connected, topological space admitting a universal cover $\tilde{X} \rightarrow X$ and $\mathfrak{a} \in H^1(X; \mathbf{R})$ is a cohomology class represented by a singular one-cocycle α . Let $x \in X$ be a reference point. Define a real valued two-cocycle $\mathfrak{G}_{x,\alpha}$ on the group $\text{Homeo}(X, \mathfrak{a})$ of homeomorphisms of X preserving the class \mathfrak{a} by the following formula

$$\mathfrak{G}_{x,\alpha}(g, h) := \int_{\gamma} g^* \alpha - \alpha$$

where γ is a path from x to hx .

Lemma 3.1.

(1) *If h and g are homeomorphisms preserving $\mathfrak{a} = [\alpha]$ then*

$$\mathfrak{G}_{x,\alpha}(g, h) = \mathfrak{K}_\alpha(g)(hx) - \mathfrak{K}_\alpha(g)(x).$$

(2) *The value $\mathfrak{G}_{x,\alpha}(g, h)$ does not depend on the choice of a path from x to hx .*

(3) *The function $\mathfrak{G}_{x,\alpha}$ is a two-cocycle on $\text{Homeo}(X, \mathfrak{a})$. That is it satisfies the following identity:*

$$\mathfrak{G}_{x,\alpha}(h, k) - \mathfrak{G}_{x,\alpha}(gh, k) + \mathfrak{G}_{x,\alpha}(g, hk) - \mathfrak{G}_{x,\alpha}(g, h) = 0.$$

(4) *The cohomology class of the cocycle $\mathfrak{G}_{x,\alpha}$ depends neither on the choice of the reference point x nor on the choice of the cocycle α (only on the cohomology class \mathfrak{a}).*

(5) *If either g preserves α or h preserves x then $\mathfrak{G}_{x,\alpha}(g, h) = 0$. \square*

Proof. For the sake of consistency we prove part 1 of the lemma, leaving the other, straightforward items, which will not be used in the paper, to the reader.

It is an immediate consequence the definition of the cocycle \mathfrak{K}_α . Indeed, we have

$$\mathfrak{G}_{x,\alpha}(g, h) = \int_x^{hx} g^* \alpha - \alpha = \int_x^{hx} \delta(\mathfrak{K}_\alpha(g)) = \mathfrak{K}_\alpha(g)(hx) - \mathfrak{K}_\alpha(g)(x).$$

\square

In what follows, as the one-cocycle α is fixed in this section, we would write \mathfrak{G}_x instead $\mathfrak{G}_{x,\alpha}$ for short.

3.B. Quasimorphisms. Let $q: G \rightarrow \mathbf{R}$ be a map defined on a group G . The **defect** $D(q)$ of the map q is defined to be

$$D(q) := \sup_{g, h \in G} |q(g) - q(gh) + q(h)|.$$

If the defect of q is finite then q is called a **quasimorphism**. A quasimorphism q is called **homogeneous** if $q(g^n) = nq(g)$ for all $n \in \mathbf{Z}$ and $g \in G$. For every quasimorphism q the formula

$$\hat{q}(g) := \lim_{n \rightarrow \infty} \frac{q(g^n)}{n}$$

defines a homogeneous quasimorphism called the homogenisation of q . Moreover, $|\hat{q}(g) - q(g)| \leq D$ for all $g \in G$ [3, Lemma 2.21]. Thus q is unbounded if and only if so is its homogenisation.

Proposition 3.2. *Let $\alpha \in H^1(X; \mathbf{R})$. Let $G \subseteq \text{Homeo}(X, \alpha)$ be a subgroup on which the cocycles \mathfrak{G}_x and \mathfrak{G}_y are bounded, for some $x, y \in X$. Then the map $q: G \rightarrow \mathbf{R}$ defined by*

$$q(g) := \mathfrak{K}_\alpha(g)(y) - \mathfrak{K}_\alpha(g)(x)$$

is a quasimorphism on G and $D(q) \leq \|\mathfrak{G}_x - \mathfrak{G}_y\| \leq \|\mathfrak{G}_x\| + \|\mathfrak{G}_y\|$, where $\|\cdot\|$ denotes the supremum norm of a bounded function.

Proof. This is a straightforward computation using the cocycle identity for \mathfrak{K}_α .

$$\begin{aligned} q(f) - q(fg) + q(g) &= \mathfrak{K}_\alpha(f)(y) - \mathfrak{K}_\alpha(f)(x) \\ &\quad - (\mathfrak{K}_\alpha(f)(gy) + \mathfrak{K}_\alpha(g)(y) - \mathfrak{K}_\alpha(f)(gx) - \mathfrak{K}_\alpha(g)(x)) \\ &\quad + \mathfrak{K}_\alpha(g)(y) - \mathfrak{K}_\alpha(g)(x) \\ &= \mathfrak{K}_\alpha(f)(gx) - \mathfrak{K}_\alpha(f)(x) - (\mathfrak{K}_\alpha(f)(gy) - \mathfrak{K}_\alpha(f)(y)) \\ &= \mathfrak{G}_x(f, h) - \mathfrak{G}_y(f, g). \end{aligned}$$

□

Example 3.3. In this example we show that the boundedness of \mathfrak{G}_x depends on the choice of a point $x \in X$. Let $X = \mathbf{R}/\mathbf{Z} \times \mathbf{R} \cup \{\infty\}$ be the two-dimensional torus. Let α be a singular one-cocycle defined by

$$\int_\gamma \alpha := \tilde{\gamma}(1) - \tilde{\gamma}(0),$$

where $\tilde{\gamma}: [0, 1] \rightarrow \mathbf{R}$ is a lift of the composition of γ followed by the projection onto \mathbf{R}/\mathbf{Z} . Let α be the class of α .

Let $g \in \text{Homeo}(X, \alpha)$ be a homeomorphism defined by

$$g(t, x) := (t + |x + 1| - |x|, x + 1).$$

Then $\mathfrak{K}_\alpha(g^n)(t, x) = |x + n| - |x|$ and it follows that

$$\begin{aligned} \mathfrak{G}_{(0,0)}(g^m, g^n) &= \mathfrak{K}_\alpha(g^m)(g^n(0, 0)) - \mathfrak{K}_\alpha(g^m)(0, 0) \\ &= |m + n| - |n| - |m|. \end{aligned}$$

This shows that $\mathfrak{G}_{(0,0)}$ is unbounded (in fact, the cocycle $\mathfrak{G}_{(t,x)}$ is unbounded whenever x is finite). On the other hand, g acts trivially on the circle $\mathbf{R}/\mathbf{Z} \times \{\infty\}$ and hence $\mathfrak{G}_{(t,\infty)} = 0$. \diamond

Example 3.4. If g is a time-one map of a gradient flow then $\mathfrak{G}_{x,\alpha}$ is bounded at every x and the local rotation number of g is equal to zero. \diamond

Theorem 3.5. Let $\alpha \in H^1(X; \mathbf{R})$ and let $g \in \text{Homeo}(X, \alpha)$ and assume that X is compact. Suppose that for some points $x, y \in X$ the cocycles \mathfrak{G}_x and \mathfrak{G}_y are bounded on the cyclic subgroup $\langle g \rangle \subset \text{Homeo}(X, \alpha)$ generated by g . If the above quasimorphism $q: \langle g \rangle \rightarrow \mathbf{R}$ is unbounded then g is undistorted in $\text{Homeo}(X, \alpha)$.

Remark 3.6. It is often the case that to prove that an element g is undistorted in a group G one constructs a homogeneous quasimorphism $q: G \rightarrow \mathbf{R}$ such that $q(g) \neq 0$. Constructing such a quasimorphism is in general very difficult. The advantage of the above theorem is that we only need to check that a naturally defined quasimorphism on a cyclic group is unbounded.

Proof of Theorem 3.5. Let Γ be a finitely generated subgroup of $\text{Homeo}(X, \alpha)$ containing g . Let $\hat{q}: \langle g \rangle \rightarrow \mathbf{R}$ be the homogenisation of the quasimorphism q . The following calculation of the translation length of g shows that g is undistorted in Γ .

$$\begin{aligned} C \cdot \tau(g) &= \lim_{n \rightarrow \infty} \frac{C \cdot |g^n|}{n} \\ &\geq \lim_{n \rightarrow \infty} \frac{\|g^n\|_\alpha}{n} \\ &\geq \lim_{n \rightarrow \infty} \frac{|q(g^n)|}{n} \\ &= |\hat{q}(g)| > 0. \end{aligned}$$

Since Γ is arbitrary, the element g is undistorted in $\text{Homeo}(X, \alpha)$. \square

Notice that Theorem 3.5 also implies Corollary 1.2. Recall that we need to prove that if x and y are fixed points of g and $\int_Y g^* \alpha - \alpha \neq 0$ then g is undistorted in $\text{Homeo}(X, \alpha)$.

First, observe that the cocycles \mathfrak{G}_x and \mathfrak{G}_y vanish identically on the cyclic group $\langle g \rangle$ because x and y are fixed points of g . By Proposition 3.2 the defect of q is zero (since it is bounded by $\|\mathfrak{G}_x\| + \|\mathfrak{G}_y\| = 0$) and we obtain that $q: \langle g \rangle \rightarrow \mathbf{R}$ is a homomorphism of groups. Furthermore

$$q(g) = \mathfrak{K}_\alpha(g)(y) - \mathfrak{K}_\alpha(g)(x) = \int_Y g^* \alpha - \alpha \neq 0$$

according to the hypothesis. Therefore q is unbounded and the statement follows from Theorem 3.5.

3.C. Local rotation number. In what follows we are interested in bounded cohomology of a group with the integer coefficients; see Gromov [12] and Monod [15] for a background on bounded cohomology.

We assume that $\alpha \in H^1(X; \mathbb{Z})$ and α is an integer-valued one-cocycle on X . Therefore $\mathfrak{G}_{x,\alpha}$ is an integer-valued two-cocycle on $\text{Homeo}(X, \alpha)$.

Example 3.7. (Ghys [11, Section 6.3]) The second bounded cohomology $H_b^2(\mathbb{Z}; \mathbb{Z})$ of the integers with integer coefficients is isomorphic to \mathbb{R}/\mathbb{Z} . To see this let $\mathfrak{c}: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ be a bounded two-cocycle. As an ordinary cocycle it is a coboundary since the second group cohomology of the group of integers is trivial. If $\mathfrak{c} = \delta \mathfrak{b}$ then, since \mathfrak{c} is bounded, the cochain \mathfrak{b} is a quasimorphism. The homogenisation of \mathfrak{b} (which is a real cochain in general) is given by $\hat{\mathfrak{b}}(n) = rn$ for some real number $r \in \mathbb{R}$. The required isomorphism

$$H_b^2(\mathbb{Z}; \mathbb{Z}) \rightarrow \mathbb{R}/\mathbb{Z}$$

is defined by $[\mathfrak{c}] \mapsto r + \mathbb{Z}$. ◇

Let $g \in \text{Homeo}(X, \alpha)$ and let $x \in X$ be a point for which the cocycle $\mathfrak{G}_{x,\alpha}$ is bounded on the cyclic group generated by g . The cohomology class

$$\mathbf{rot}_{x,\alpha}(g) = [\mathfrak{G}_{x,\alpha}] \in H_b^2(\langle g \rangle; \mathbb{Z}) = \mathbb{R}/\mathbb{Z}$$

is called the **local rotation number** of g at the point $x \in X$.

Let us explain the geometry of the local rotation number. Take a path $\eta_{x,1}: [0, 1] \rightarrow X$ from x to gx and let $\eta_{x,n}$ be the concatenation of paths $g^k(\eta_{x,1})$ for k ranging from 0 to $n-1$. Define a map $\mathfrak{b}_x: \langle g \rangle \rightarrow \mathbb{R}$ by

$$(3.8) \quad \mathfrak{b}_x(g^n) := - \int_{\eta_{x,n}} \alpha.$$

Observe that $\delta \mathfrak{b}_x = \mathfrak{G}_x$ on the cyclic group $\langle g \rangle$. Since \mathfrak{G}_x is bounded on $\langle g \rangle$ we get that \mathfrak{b}_x is a quasimorphism and that its homogenisation satisfies $\hat{\mathfrak{b}}_x(g^n) = r_x(g)n$ for a suitable representative of the local rotation number of g at x . This shows that there exists a constant $C_x > 0$ such that

$$|\mathfrak{b}_x(g^n) - r_x(g)n| \leq C_x$$

for all $n \in \mathbb{Z}$. We thus obtain that

$$(3.9) \quad \lim_{n \rightarrow \infty} \frac{\mathfrak{b}_x(g^n)}{n} = r_x(g)$$

and hence the fractional part of above limit represents the local rotation number of g at x :

$$\mathbf{rot}_{x,\alpha}(g) = r_x(g) + \mathbb{Z}.$$

Indeed, since α has integral periods, the dependence of $b_x(g^n)$ on the choice of the path $\eta_{x,1}$ is up to an integer constant only. This implies that the above computation of the local rotation number does not depend on the choice of a path $\eta_{x,1}$. The next result immediately follows from the above discussion.

Proposition 3.10. *Let X be a smooth compact manifold and let $p: X_{\mathfrak{a}} \rightarrow X$ be the cyclic covering associated with $\mathfrak{a} \in H^1(X; \mathbf{Z})$. Assume that \mathfrak{a} is represented by a closed smooth one-form α . Let $F: X_{\mathfrak{a}} \rightarrow \mathbf{R}$ be a smooth function such that $dF = p^*\alpha$. Then*

$$\text{rot}_{x,\alpha}(g) = - \lim_{n \rightarrow \infty} \frac{F(\tilde{g}^n(\tilde{x}))}{n} + \mathbf{Z}$$

provided that the limit exists. \square

3.D. Proof of Theorem 1.6. In order to apply Theorem 3.5 we need to prove that the quasimorphism $q: \langle g \rangle \rightarrow \mathbf{R}$ from Proposition 3.2 is unbounded.

Let $\gamma, \eta_{x,n}, \eta_{y,n}: [0, 1] \rightarrow X$ be paths from x to y , x to $g^n x$ and y to $g^n y$ respectively and $n \in \mathbf{Z}$. As above assume that $\eta_{x,n}$ is a concatenation of the paths $g^k(\eta_{x,1})$ for k ranging from 0 to $n-1$ and similarly for $\eta_{y,n}$.

Let $b_x, b_y: G \rightarrow \mathbf{R}$ and $r_x(g), r_y(g) \in \mathbf{R}$ be defined as in 3.8 and 3.9. Let \square_n be a concatenation of $-\gamma$, $\eta_{x,n}$, $g^n \gamma$ and $-\eta_{y,n}$. We get the following computation.

$$\begin{aligned} q(g^n) &= \int_{\square_n} (g^n)^* \alpha - \alpha \\ &= \int_{\square_n} \alpha - \int_{\eta_{x,n}} \alpha + \int_{\eta_{y,n}} \alpha \\ &= n \int_{\square_1} \alpha + b_x(g^n) - b_y(g^n) \\ &= n \left(\int_{\square_1} \alpha + (r_x(g) - r_y(g)) \right) + O(1). \end{aligned}$$

Since α has integral periods and the difference $r_x(g) - r_y(g)$ is not an integer by the hypothesis, we get that the quasimorphism q is unbounded. \square

4. APPENDIX: ON SINGULAR ONE-COCYCLES

The results of this section are used to prove Lemma 2.1.

Let \mathbf{A} be an Abelian group which is a trivial coefficient system over X . Since $H^1(X; \mathbf{A}) = \text{Hom}(\pi_1(X), \mathbf{A})$ one can define a cover

$$\mathbf{A} \rightarrow X_{\mathfrak{a}} := \tilde{X} \times_{\pi_1(X)} \mathbf{A} \rightarrow X,$$

where \tilde{X} is the universal cover of X and $\pi_1(x)$ acts on \mathbf{A} via homomorphism defined by \mathfrak{a} . In what follows, the action $\mathbf{A} \times X_{\mathfrak{a}} \rightarrow X_{\mathfrak{a}}$ by the deck transformations will be denoted additively: $(a, z) \mapsto a + z$.

Let $x \in X$ be a reference point in X and let $\tilde{x} \in p^{-1}(x)$ be a reference point in $X_{\mathfrak{a}}$. Let α be a singular cocycle representing the class \mathfrak{a} . That is, α is a homomorphism $C_1(X; \mathbf{A}) \rightarrow \mathbf{A}$ defined on the group of chains on X with the coefficients in \mathbf{A} . It defines an \mathbf{A} -equivariant map $\mathbf{a}: X_{\mathfrak{a}} \rightarrow \mathbf{A}$ in the following way. Given a point $\tilde{y} \in p^{-1}(y)$ let $\gamma: [0, 1] \rightarrow X$ be a path from x to y . Let $\tilde{\gamma}: [0, 1] \rightarrow X_{\mathfrak{a}}$ be its lift such that $\tilde{\gamma}(0) = \tilde{x}$. Then we define $\mathbf{a}(\tilde{y})$ as the unique element such that $\int_{\gamma} \alpha + \tilde{y} = \mathbf{a}(\tilde{y}) + \tilde{\gamma}(1)$. If we put $\tilde{y} := \tilde{\gamma}(1)$ we obtain that

$$\mathbf{a}(\tilde{\gamma}(1)) = \int_{\gamma} \alpha.$$

Let us check that \mathbf{a} does not depend on the choice of the path γ . Let γ_{\pm} be two paths from x to y and let \mathbf{a}_{-} and \mathbf{a}_{+} denote the corresponding maps. By letting $\tilde{y} = \tilde{\gamma}_{+}(1)$ in the equality

$$\int_{\gamma_{+}} \alpha + \tilde{\gamma}_{-}(1) = \int_{\gamma_{-}} \alpha + \tilde{\gamma}_{+}(1)$$

we get

$$\int_{\gamma_{+}} \alpha + \tilde{\gamma}_{-}(1) = \int_{\gamma_{-}} \alpha + \tilde{y}$$

which shows that $\mathbf{a}_{+}(\tilde{y}) = \int_{\gamma_{+}} \alpha = \mathbf{a}_{-}(\tilde{y})$ as claimed.

The equivariance of \mathbf{a} is immediate from the definition. Another choice of a reference point results in changing \mathbf{a} by an additive constant.

Let $\mathbf{a}: X_{\mathfrak{a}} \rightarrow \mathbf{A}$ be an \mathbf{A} -equivariant function. Let $\gamma: [0, 1] \rightarrow X$ be a path and let $\tilde{\gamma}: [0, 1] \rightarrow X_{\mathfrak{a}}$ be its lift. The following formula defines a singular one-cocycle with values in \mathbf{A} .

$$\int_{\gamma} \alpha = \mathbf{a}(\tilde{\gamma}(1)) - \mathbf{a}(\tilde{\gamma}(0))$$

Lemma 4.1. *The above constructions are inverse to each other and hence provide a bijective correspondence between singular one-cocycles in the class $\mathfrak{a} \in H^1(X, \mathbf{A})$ and \mathbf{A} -equivariant maps $\mathbf{a}: X_{\mathfrak{a}} \rightarrow \mathbf{A}$ up to the constants.*

Proof. Let α be a singular one-cocycle representing the class \mathfrak{a} . It defines an equivariant map $\mathbf{a}: X_{\mathfrak{a}} \rightarrow \mathbf{A}$ such that $\int_{\gamma} \alpha + \tilde{y} = \mathbf{a}(\tilde{y}) + \tilde{\gamma}(1)$ for every path $\gamma: [0, 1] \rightarrow X$ from x to y . We need to check that $\int_{\gamma} \alpha = \mathbf{a}(\tilde{\gamma}(1)) - \mathbf{a}(\tilde{\gamma}(0))$.

Let $\tilde{y} := \tilde{\gamma}(1)$ where the lift $\tilde{\gamma}$ is chosen so that $\mathbf{a}(\tilde{\gamma}(0)) = 0$. Then

$$\int_{\gamma} \alpha + \tilde{\gamma}(1) = \mathbf{a}(\tilde{\gamma}(1)) + \tilde{\gamma}(1)$$

implies that $\int_{\gamma} \alpha = \mathbf{a}(\tilde{\gamma}(1))$.

Conversely, let $\mathbf{a}: X_{\mathbf{a}} \rightarrow \mathbf{A}$ be an \mathbf{A} -equivariant map. It defines a singular cocycle α by the identity $\int_{\gamma} \alpha = \mathbf{a}(\tilde{\gamma}(1))$, where $\tilde{\gamma}$ is a lift of γ such that $\mathbf{a}(\tilde{\gamma}(0)) = 0$. We then clearly get that $\int_{\gamma} \alpha + \tilde{\gamma}(1) = \mathbf{a}(\tilde{\gamma}(1)) + \tilde{\gamma}(1)$. \square

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